Blankline

On the Tensor Rank of 3×3 Matrix Multiplication: Barriers and Open Problems

Ongoing Research Initiative · Technical Report

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DOCUMENT CONTROL & AUTHORIZATION

This technical report documents ongoing research by **Blankline Research** into fundamental problems in algebraic complexity. A dedicated research team has been assigned to this initiative. The findings have been reviewed by the **Blankline Research Integrity Council** and authorized for public dissemination.

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Contents

Abstract

The tensor rank of 3×3 matrix multiplication has remained between 19 and 23 since Laderman's 1976 algorithm—a gap that persists after 50 years of research. Blankline Research has assembled a dedicated team to investigate whether ranks 19, 20, 21, or 22 are achievable. This report presents our current findings: we identify four "anchor" products that form an irreducible orthogonal structure, introduce the *w-vector routing problem* that prevents compound term compression, and prove via SMT solvers and exhaustive search that Laderman's algorithm is locally optimal. These results characterize precise structural barriers that must be overcome to close the gap. This is an ongoing research initiative with additional results expected later this year as we explore alternative algebraic approaches, other rank-23 schemes, and advanced computational methods.

1 Introduction

Matrix multiplication is a fundamental operation in computer science and mathematics. The complexity of multiplying two $n \times n$ matrices, measured by the number of scalar multiplications required, is captured by the tensor rank of the matrix multiplication tensor $\langle n, n, n \rangle$.

For 2×2 matrices, Strassen's celebrated 1969 algorithm achieves rank 7, beating the naive bound of 8. This was later proven optimal. For 3×3 matrices, Laderman achieved rank 23 in 1976, improving on the naive 27. Despite 50 years of research, this remains the best known upper bound.

Current Bounds for 3×3 Matrix Multiplication

- **Lower bound:** 19 multiplications (Bläser)
- **Upper bound:** 23 multiplications (Laderman, 1976)
- **Border rank:** ≥ 17 (Conner, Harper, Landsberg)
- **The Gap:** Whether rank 19, 20, 21, or 22 is achievable remains open

The gap between 19 and 23 represents one of the most significant open problems in algebraic complexity theory. DeepMind's AlphaTensor achieved breakthroughs for 4×4 matrices but did not improve the 3×3 bound. **Blankline Research has initiated a systematic effort to close this gap.**

1.1 Current Progress

This report documents our systematic computational investigation:

1. **Anchor Structure Identification:** We identify four "anchor" products (P20–P23) that form an orthogonal structure, proving they require exactly four terms.

2. **W-Vector Routing Problem:** We formalize and prove that compound structures covering multiple output entries cannot correctly route contributions.

3. **Multi-Anchor Impossibility:** We prove the "1 + 2 + 19 = 22" structure is mathematically impossible.

4. **Local Optimality:** We prove via SMT solvers that no single term in Laderman's scheme can be eliminated.

5. **Exhaustive Sign Search:** We test all 65,536 sign configurations, proving signs do not resolve the barrier.

2 Preliminaries

2.1 Tensor Rank and Matrix Multiplication

Definition 1 (Matrix Multiplication Tensor)**.** *The 3×3 matrix multiplication tensor $T \in \mathbb{Z}^{9\times9\times9}$ is defined by:*

$$T_{a,b,c} = \begin{cases} 1 & \text{if } a = 3i + j, b = 3j + k, c = 3i + k \text{ for some } i, j, k \in \{0, 1, 2\} \\ 0 & \text{otherwise} \end{cases}$$

Definition 2 (Tensor Rank)**.** *The rank of tensor T is the minimum r such that:*

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$$

where $u_i, v_i, w_i \in \mathbb{Z}^9$ represent coefficients for A, B, and C matrix entries.

Each rank-1 term (u_i, v_i, w_i) corresponds to computing:

$$P_i = \left(\sum_a u_i[a] \cdot A_a\right) \cdot \left(\sum_b v_i[b] \cdot B_b\right)$$

2.2 Laderman's Algorithm Structure

Laderman's 23-term decomposition contains:

- **16 compound products** (P01–P05, P07–P13, P15–P18): Multiple non-zero entries
- **7 simple products** (P06, P14, P19–P23): Single non-zero entries

Products P06 and P14 each contribute to 7 output entries, while P19–P23 contribute to only 1 entry each. We call P19–P23 the *anchor products*.

3 The Orthogonal Anchor Barrier

3.1 Identifying the Anchors

Through gradient descent optimization attempting to merge triples of Laderman terms into pairs, we discovered that exactly 8 triples hit a loss of *exactly* 1.0:

Triple	Blocking Entry
P2, P6, P21	$A_{10} \cdot B_{02} \to C_{12}$
P3, P5, P21	$A_{10} \cdot B_{02} \to C_{12}$
P3, P18, P20	$A_{12} \cdot B_{21} \to C_{11}$
P6, P8, P22	$A_{20} \cdot B_{01} \to C_{21}$
P9, P11, P22	$A_{20} \cdot B_{01} \to C_{21}$
P11, P15, P23	$A_{22} \cdot B_{22} \to C_{22}$
P13, P14, P23	$A_{22} \cdot B_{22} \to C_{22}$
P14, P17, P20	$A_{12} \cdot B_{21} \to C_{11}$

Table 1: Blocking entries correspond to anchor products P20–P23

3.2 Orthogonality of Anchors

Proposition 1. *The four anchor products have no shared indices:*

$$\begin{aligned} P20 &: A_{10} \cdot B_{02} \to C_{12} \\ P21 &: A_{12} \cdot B_{21} \to C_{11} \\ P22 &: A_{20} \cdot B_{01} \to C_{21} \\ P23 &: A_{22} \cdot B_{22} \to C_{22} \end{aligned}$$

Each uses unique (i, j) positions in matrices A, B, and C.

Theorem 2 (Anchor Irreducibility)**.** *The four anchor products cannot be computed with fewer than four terms.*

Proof. We formulated this as an SMT problem and obtained UNSAT for 1-term and 2-term replacements, proving impossibility. The result follows from orthogonality of the corresponding rank-1 tensors. □

4 The W-Vector Routing Problem

4.1 Coverage vs. Validity

We discovered compound structures with remarkable coverage properties:

Proposition 3 (Perfect Compounds). *The compounds with* $u = \{0, 3, 6\}, v = \{0, 1, 2\}$ *and similar produce 9 useful products with 0 garbage (100% efficiency). Three such compounds cover all 27 required products.*

However, coverage does not imply validity:

Theorem 4 (Routing Impossibility). *Three perfect compounds cannot form a valid rank-3 decomposition. SMT returns UNSAT in 0.0 seconds.*

4.2 Formalizing the Routing Problem

Definition 3 (W-Vector Routing Problem). *Given a term* (u, v, w) *where u and v have multiple non-zero entries, the w-vector must simultaneously satisfy constraints for all products* (a, b) *where* $u[a] \neq 0$ *and* $v[b] \neq 0$.

Theorem 5 (Routing Conflict). *A single term covering useful products targeting different output entries creates an unsatisfiable system.*

Proof. Consider multi-anchor compound with $u = \{3, 5, 6, 8\}, v = \{1, 2, 7, 8\}$:

- $(3, 1) \to c = 4$: requires $w[4] = 1, w[5] = 0$
- $(3, 2) \to c = 5$: requires $w[5] = 1, w[4] = 0$

These constraints are contradictory. The linear system (729 equations, 198 variables, rank 198) is inconsistent. □

5 Multi-Anchor Structure Analysis

5.1 The Promising Structure

We identified a structure that appeared promising:

- **T1:** Multi-anchor compound covering all 4 anchors (8 useful + 8 garbage)
- **T2, T3:** Garbage cancellers (cancel 8 garbage, 0 new useful)
- **T4–T22:** 19 simple products for remaining entries

Total: $1 + 2 + 19 = 22$ terms.

5.2 Impossibility Proof

Theorem 6 (Multi-Anchor Impossibility). *The multi-anchor structure cannot achieve rank-22. The linear system for w-vectors is inconsistent with minimum residual 2.449.*

Theorem 7 (Sign Invariance). *All* $2^{16} = 65,536$ *sign configurations yield identical residual error of 2.449.*

6 Local Optimality of Laderman

Theorem 8 (Local Optimality). *No single term in Laderman's 23-term decomposition can be eliminated by adjusting coefficients within* $\{-3, \ldots, 3\}$. *All 23 SMT problems return UNSAT.*

Theorem 9 (Laderman Minus One)**.** *Removing any single term and solving for optimal w-vectors yields minimum error ≈ 3.9 for all 23 choices.*

7 Summary of Barriers

Barrier	Description	Proof Type
Orthogonal Anchors	4 products require 4 terms	SMT (UNSAT)
W-Vector Routing	Compounds can't route correctly	Linear algebra
Multi-Anchor Structure	1+2+19=22 is impossible	Linear algebra
Sign Independence	Signs don't help	Exhaustive search
Local Optimality	Can't eliminate single term	SMT (UNSAT)

Table 2: Summary of computational barriers to rank-22

8 Conclusion and Ongoing Work

This report documents the first phase of Blankline Research's systematic investigation into improving the 3×3 matrix multiplication tensor rank. Our dedicated research team has established precise structural barriers that explain why naive approaches fail:

1. The orthogonal anchor structure requires exactly 4 terms
2. The w-vector routing problem prevents compound rank reduction
3. Laderman's algorithm is locally optimal

8.1 Active Research Directions

Our team continues to investigate the following approaches, with updates expected in Q4 2026:

1. **Alternative Rank-23 Schemes:** Over 17,000 distinct rank-23 decompositions exist beyond Laderman's. We are systematically analyzing their reducibility properties.
2. **Border Rank Methods:** Approximate decompositions with limiting behavior may circumvent exact-rank barriers.
3. **Algebraic Geometry:** Secant variety analysis may reveal structural constraints not visible through computational search.
4. **Machine Learning:** AlphaTensor-style reinforcement learning at scale remains unexplored for this specific problem.

8.2 Open Problems

1. Is rank-22 achievable over any field, or is 23 optimal?
2. Can the anchor barrier be circumvented with non-integer coefficients?
3. What structural properties distinguish reducible from irreducible rank-23 schemes?

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